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Discrete and continuous graded contractions of Lie algebras and superalgebras

M de Montigny and J Patera
Centre de recherches mathématiques, Université de Montréal, CP 6128–A, Montréal (Québec) H3C 3J7, Canada

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Abstract. Grading preserving contractions of Lie algebras and superalgebras of any type over the complex number field are defined and studied. Such contractions fall naturally into two classes: the Wigner–Inönü-like continuous contractions and new discrete contractions. A general method is described for any Abelian grading semigroup and any Lie algebra or superalgebra admitting such a grading. All contractions preserving $\mathbb{Z}_2$-, $\mathbb{Z}_3$-, and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings are found. Examples of these gradings and contractions for the simple Lie algebra A_2, affine Kac–Moody algebra $A_1^{(1)}$ and the simple superalgebra osp$(2,1)$ are shown.

1. Introduction

The interest of physicists in contracting Lie algebras is well demonstrated in the literature [1, and references therein]. It stems from the need to relate, in a meaningful way, the symmetry Lie groups (Lie algebras) of different physical systems with one another and thus to bring the corresponding phenomena to a common understanding. The simplest of such relations is that associated with symmetry breaking where two Lie algebras are related by a homomorphism (inclusion).

The Wigner–Inönü contraction of a Lie algebra L to L^ϵ is an example of a relation which is not, in general, a homomorphism. (For other non-homomorphic relations see, for example, [2–4].) Graded contractions as defined in this article allow many more contraction parameters to be introduced and consequently a much larger variety of contraction 'limits' to be studied.

The defining feature of our method is the preservation of a chosen grading during the contraction. A grading of a Lie algebra L implies the subspace decomposition (2.1) and the commutation relations (2.2) described later. Two gradings of L are equivalent if their grading decompositions differ by a transformation from the group of automorphisms of L. During a general grading-preserving contraction all the elements *of a grading subspace L_j are treated the same way: only the commutation relations* between whole subspaces are modified. It turns out that such an approach offers an efficient tool for computing in comparison with the general theory of deformations of Lie algebras (cf [5, 6 and references therein]). In particular, one needs neither to fix the dimension of the algebra (it can be finite or infinite), nor to make a distinction between a Lie algebra and superalgebra. Moreover, the preservation of the grading can also be made into a natural guiding principle in defining and studying contractions of representations [7].

The purpose of this article is to generalize the traditional Wigner–Inönü contractions [1] of Lie algebras in the following directions:

(1) The $\mathbb{Z}_2$-grading is replaced by any Abelian semigroup as the grading semigroup.

(2) In addition to the continuous limits, we define and also find discrete contractions in all but the simplest cases.

(3) For any fixed grading semigroup, the problem is solved simultaneously for Lie algebras and superalgebras of any type and dimension provided they admit the chosen grading.

The precise relation between graded contractions of a general Lie algebra and its deformations has yet to be established. However, it appears that any deformation of a simple Lie algebra over the complex number field preserves some grading of the algebra, hence it is a graded contraction.

An investigation of all possible (graded) contractions of a given Lie algebra and superalgebra hinges on the knowledge of all its gradings. Even for Lie algebras of modest dimension like $\mathrm{sl}(3,\mathbb{C})$ this becomes a major problem [8] even if in this case all gradings are known [9]. Simply there are many possible graded contractions. The general question of the systematic study of gradings of a Lie algebra has apparently been raised only recently [10]. Nevertheless it has already stimulated the present work, as well as [2, 3, 7–9, 11].

Technically, the difference between the traditional method of contracting Lie algebras and its present generalization is best shown in the way the Jacobi identity (after a contraction) is enforced. The traditional parametrization automatically guarantees the validity of the Jacobi identities. In our case the identities impose a system of quadratic relations on the contraction parameters. The number of contraction parameters in the traditional approach cannot exceed the order of the grading group, in our case it grows quadratically with the order.

The discrete contractions conceptually differ from the traditional continuous contractions [1] or, more generally, deformations of Lie algebras [5, 6]. In spite of that they arise here 'naturally', being (non-trivial) solutions of the same system of equations (2.17) which gives all the graded continuous contractions as a limit of its rather trivial solutions. Furthermore, the set consisting of the discrete and continuous contractions is closed under the contraction composition law (2.8). Moreover, in certain rather typical cases an outcome of a discrete contraction may be a parameter dependent family of Lie algebras. For certain limit values of the parameter one gets a continuous contraction (cf (4.19) as $p \longmapsto 1$), i.e. there is a continuum of discrete contractions which is 'infinitesimally' close to a continuous contraction.

All Lie algebras and superalgebras are taken here over the complex number field. Analogous problems for the real field will be considered elsewhere. In general, it is straightforward to decide whether a given grading of a complex algebra is also a grading of its chosen real form and, consequently, whether the graded contractions also apply to that real case.

The article is organized as follows. In section 2 the general method is described. In the subsequent three sections the method is applied to an arbitrary Lie algebra or superalgebra which admits $\mathbb{Z}_2$-, $\mathbb{Z}_3$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings. The contractions for the generic case are found; the non-generic cases are obtained by further restriction of the generic ones. The last three sections contain examples of $\mathbb{Z}_2$-, $\mathbb{Z}_3$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded contractions of the simple Lie algebra $\mathrm{sl}(3,\mathbb{C})$, the affine Kac–Moody algebra $A_1^{(1)}$, simple superalgebra $\mathrm{osp}(2,1)$ and the simple Lie algebra $\mathrm{sl}(2,\mathbb{C})$. The purpose of the sl(2, $\mathbb{C}$) example in sections 6–8 is to illustrate the discrete contractions (in section 7) on the simplest case possible.

More information concerning superalgebras and affine Kac–Moody algebras is found for example in [12–14].

2. Graded contractions of Lie algebras and superalgebras

In this section we describe our method in general. It allows us to find the contractions which preserve a chosen fixed grading of a Lie algebra or superalgebra. In fact at no point in this section do we need to distinguish between the two types of algebras. There is no restriction as to the dimension of the algebra, it may be finite or infinite. We require no features other than the presence of the chosen grading.

For simplicity of presentation we continually speak about Lie algebras and commutators, only occasionally underlining the fact that the superalgebras with the same type of grading are being considered as well. Similarly we speak about grading group rather than semigroup.

Suppose L is a Lie algebra graded by an Abelian finite group G. That means we have the grading decomposition

$$L = \bigoplus_{j \in G} L_j \tag{2.1}$$

of L into the direct sum of grading subspaces L_j, and the commutators of the subspaces satisfy the relations

$$[L_j, L_k] \subseteq L_{j+k} \qquad j, k, j+k \in G. \tag{2.2}$$

Here we are using the additive notation for the multiplication in G. Let us emphasize that $[L_j, L_k]$ is a linear space generated by commuting every element of L_j with every element of L_k. Therefore it coincides with the one denoted by $[L_k, L_j]$. Similarly, in the case of superalgebras, it is immaterial whether such a space is obtained as a result of commutation or anticommutation.

In general we shall not be interested in the details of the structure of L (except for some examples), but we will need to specify whether a commutator $[L_j, L_k]$ is identically zero or not. This information is conveniently, for our purposes, provided by the symmetric matrix $\kappa = (\kappa_{ij})$:

$$\kappa_{jk} = \begin{cases} 0 & \text{if } [L_j, L_k] = 0 \\ 1 & \text{if } [L_j, L_k] \neq 0. \end{cases} \tag{2.3}$$

Hence without loss of generality we can write

$$[L_j, L_k] \subseteq \kappa_{jk} L_{j+k} \tag{2.4}$$

and speak of a Lie algebra L with the G-graded structure κ. Whenever necessary we indicate the G-graded structure of L explicitly by writing L^κ for L.

Let us now define the G-graded contractions. Suppose a G-graded Lie algebra L^κ of (2.1) and (2.4) is given.

A G-graded contraction L^{ε} of L^{κ}:

$$L^{\kappa} \xrightarrow{\gamma} L^{\kappa \bullet \gamma} = L^{\varepsilon} \tag{2.5}$$

is the Lie algebra L^{ε} with the grading decomposition (2.1) isomorphic to that of L^{κ} and with the contracted commutators $[\ ,\ ]_{\varepsilon}$,

$$[L_j, L_k]_{\varepsilon} = \gamma_{jk}[L_j, L_k] \subseteq \gamma_{jk}\kappa_{jk}L_{j+k} = \varepsilon_{jk}L_{j+k} \tag{2.6}$$

given by the commutators in L^{κ} and the contraction parameters $\varepsilon_{jk} \in \mathbb{C}$.

The contraction is determined by the matrices $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$ or, equivalently, by $\boldsymbol{\varepsilon}$. The matrix $\boldsymbol{\gamma}$ coincides with $\boldsymbol{\varepsilon}$ in the case of L^{κ} with all matrix elements of $\boldsymbol{\kappa}$ equal to 1. We say then that $\boldsymbol{\kappa}$ is of *generic* type and write $\boldsymbol{\kappa} = (1)$.

Note that the matrices $\boldsymbol{\kappa}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\varepsilon}$ are by definition symmetric with respect to transposition

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}^{\mathrm{T}} \qquad \boldsymbol{\gamma} = \boldsymbol{\gamma}^{\mathrm{T}} \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{T}}. \tag{2.7}$$

In order that L^{ε} is a Lie algebra, the matrix $\boldsymbol{\varepsilon}$ of the contraction parameters (*contraction matrix* or just *contraction* for short) must not violate the Jacobi identity. A large part of the rest of the article is devoted to the analysis of this requirement.

The relations (2.6) can be used to motivate the introduction of an uncommon matrix composition rule for $\boldsymbol{\kappa}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\varepsilon}$. Namely,

$$\boldsymbol{\varepsilon} = \boldsymbol{\kappa} \bullet \boldsymbol{\gamma} = \boldsymbol{\gamma} \bullet \boldsymbol{\kappa} \tag{2.8}$$

defined in terms of matrix elements as

$$\varepsilon_{jk} = \gamma_{jk}\kappa_{jk} \tag{2.9}$$

with no summations implied.

The matrix $\boldsymbol{\varepsilon}$ defined in (2.8) and (2.9) gives a possible contraction of L. Most of the values of its non-zero matrix elements can often be restricted to, say, 1 by renormalization (see (2.10) below) of the grading subspaces of the contracted algebra, without changing its isomorphy class. Whether or not such a renormalization is possible needs to be investigated for each case separately.

A special case of the composition rule (2.8) arises naturally when the commutation relation (2.4) is modified by renormalization of the bases of the subspaces by arbitrary non-zero constants,

$$L_j \longrightarrow a_j L_j \qquad j \in G, \quad 0 \leq a_j \in \mathbb{C}. \tag{2.10}$$

Then we have

$$[L_j, L_k] \subseteq \frac{a_j a_k}{a_{j+k}} \kappa_{jk} L_{j+k} = \varepsilon_{jk} L_{j+k} \tag{2.11}$$

where

$$\boldsymbol{\varepsilon} = \boldsymbol{\gamma} \bullet \boldsymbol{\kappa} \tag{2.12}$$

with γ given by

$$\gamma = (\gamma_{jk}) = \left(\frac{a_j a_k}{a_{j+k}}\right). \tag{2.13}$$

Hence such a γ determines a contraction.

We say that a contraction is *trivial* if either $L^{\boldsymbol{\varepsilon}}$ is isomorphic to $L^{\boldsymbol{\kappa}}$, or if $L^{\boldsymbol{\varepsilon}}$ is Abelian, i.e. every $\varepsilon_{jk} = 0$. We write then $\boldsymbol{\varepsilon} = (0)$.

A non-trivial contraction is either *continuous* or *discrete* according to whether or not it can be achieved by a continuous change

$$1 \longrightarrow \gamma_{jk} \qquad \text{for all } j,k \in G,\ \gamma_{j,k} \in \mathbb{C} \tag{2.14}$$

without ever violating the Jacobi identity with the intermediate values. Since during a continuous change (2.14) the Jacobi identity must not be affected, its validity has to be independent of the values of γ_{jk}s. That happens precisely if each γ_{jk} is of the form (2.13), including the limit values of 0.

The task of determining the non-trivial contractions of $L^{\boldsymbol{\varepsilon}}$ of $L^{\boldsymbol{\kappa}}$ amounts to determining the matrices $\boldsymbol{\varepsilon}$ which do not violate the Jacobi identities. Since, in general, there are many different G-graded structures $\boldsymbol{\kappa}$ possible, the problem should be solved for each of them. Some of those cases will be simpler than others depending on the number of zeros in $\boldsymbol{\kappa}$, the most complicated being the generic case of $\boldsymbol{\kappa}$ with no zeros at all. The latter case corresponds to $\boldsymbol{\varepsilon} = \boldsymbol{\gamma}$. Suppose we have found all non-trivial $\boldsymbol{\gamma}$ in the generic case. Then using them in (2.8) we find the desired $\boldsymbol{\varepsilon}$ for any non-generic $\boldsymbol{\kappa}$, although different $\boldsymbol{\gamma}$ do not necessarily yield different $\boldsymbol{\varepsilon}$ and the actual ranges of non-zero matrix elements of $\boldsymbol{\varepsilon}$ need to be investigated separately.

Let us now describe a method of finding the matrices $\boldsymbol{\gamma}$ for the generic case, i.e. $\boldsymbol{\varepsilon} = \boldsymbol{\gamma}$. The Jacobi identities by definition must hold for the Lie algebra before a contraction. Without loss of generality it suffices for our purpose to require

$$[L_m,[L_j,L_k]] + [L_j,[L_k,L_m]] + [L_k,[L_m,L_j]] = 0 \tag{2.15}$$

for all $j,k,m \in G$. After a contraction, one has also, in addition to (2.15), the Jacobi identity for $L^{\boldsymbol{\gamma}}$,

$$\gamma_{jk}\gamma_{m,j+k}[L_m,[L_j,L_k]] + \gamma_{km}\gamma_{j,k+m}[L_j,[L_k,L_m]] + \gamma_{mj}\gamma_{k,m+j}[L_k,[L_m,L_j]] = 0. \tag{2.16}$$

The equalities (2.15) and (2.16) can hold simultaneously in this generic case only if one has

$$\gamma_{jk}\gamma_{m,j+k} = \gamma_{km}\gamma_{j,k+m} = \gamma_{mj}\gamma_{k,m+j}. \tag{2.17}$$

One observes that, having instead of (2.15) and (2.16), the analogous super-Jacobi identities, would result in the same equation (2.17).

Non-trivial solutions of (2.17) determine the contractions in the generic case. We solve (2.17) subsequently for specific grading groups G. Note that (2.16) is automatically satisfied when the γ_{jk} are given by (2.13) (including the limit values 0 in the numerators).

It is clear that the composition of two solutions of (2.17), say γ_1 and γ_2, according to (2.8) gives

$$\gamma = \gamma_1 \bullet \gamma_2 \tag{2.18}$$

which also is a solution of (2.17), hence a contraction.

If we had considered a Lie algebra with non-generic grading structure, some of the terms in (2.15) and, consequently, also in (2.16), would not be present. In such a case the corresponding equalities would be *absent* from the system of equations (2.17). Therefore the G-graded contractions of Lie algebras with non-generic grading structures are determined by a *subset* of equations (2.17).

For the non-generic grading structures $\boldsymbol{\kappa}$ one may use the solutions γ for the generic case and the composition rule $\bullet$ of (2.8) to generate the matrices $\boldsymbol{\varepsilon}$ of many contractions $L^{\boldsymbol{\varepsilon}}$ of $L^{\boldsymbol{\kappa}}$. In some cases (cf. (5.9)) all contractions are generated in this way. In all cases such a composition yields a contraction. For any fixed $\boldsymbol{\kappa}$, the set of contraction matrices is closed under the operation $\bullet$ up to (2.10). Numerous examples of this are shown subsequently.

It is useful to introduce the following convention: whenever $\varepsilon_{00} \neq 0$, we renormalize the basis of the grading susbspace L_0 so that $\varepsilon_{00} = 1$.

3. Contractions of $\mathbb{Z}_2$-graded Lie algebras and superalgebras

Let us consider here the cyclic group $\mathbb{Z}_2$ of two elements as the grading group. A $\mathbb{Z}_2$-graded Lie algebra decomposes as a linear space

$$L = L_0 \oplus L_1 \tag{3.1}$$

with the commutation relation in the generic case

$$0 \neq [L_j, L_k] \subseteq L_{j+k} \qquad j,\ k,\ j+k \pmod 2. \tag{3.2}$$

The general equations (2.17) specialize in this case to

$$\begin{aligned} \gamma_{00}\gamma_{11} &= \gamma_{01}\gamma_{11} \\ \gamma_{00}\gamma_{01} &= \gamma_{01}^2. \end{aligned} \tag{3.3}$$

Besides the two trivial solutions of (3.3):

$$\gamma = (1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \gamma = (0) = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \tag{3.4}$$

there are three non-trivial ones:

$$\gamma^{\mathrm{I}} = \begin{pmatrix} 1 & 1 \\ 1 & \cdot \end{pmatrix} \qquad \gamma^{\mathrm{II}} = \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix} \qquad \gamma^{\mathrm{III}} = \begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix}. \tag{3.5}$$

Only γ^{III} is a discrete contraction. Here the dot denotes the matrix element 0.

The composition rule for the contraction matrices reads as follows

$$\gamma^{\mathrm{A}} \bullet \gamma^{\mathrm{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & bf \\ cg & dh \end{pmatrix}. \tag{3.6}$$

Non-trivial contractions result from composing

$$\begin{aligned} &\gamma^{\mathrm{A}} \bullet \gamma^{\mathrm{A}} = \gamma^{\mathrm{A}} \qquad \text{for A} = \text{I, II, III} \\ &\gamma^{\mathrm{I}} \bullet \gamma^{\mathrm{III}} = \gamma^{\mathrm{III}} \end{aligned} \tag{3.7}$$

but no new contractions emerge in this way.

The graded adjoint action of L on itself,

$$\mathrm{ad}_j\, L_k = [L_j, L_k] \tag{3.8}$$

can also be described as follows:

$$\begin{pmatrix} \mathrm{ad}_0 & \mathrm{ad}_1 \\ \mathrm{ad}_1 & \mathrm{ad}_0 \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 \end{pmatrix} = \begin{pmatrix} \mathrm{ad}_0\, L_0 + \mathrm{ad}_1\, L_1 \\ \mathrm{ad}_1\, L_0 + \mathrm{ad}_0\, L_1 \end{pmatrix} = \begin{pmatrix} [L_0, L_0] + [L_1, L_1] \\ [L_0, L_1] \end{pmatrix}. \tag{3.9}$$

After a contraction one has the contracted grading action either as

$$(\mathrm{ad}_j\, L_k)_\gamma = \gamma_{jk}\, \mathrm{ad}_j\, L_k = \gamma_{jk}[L_j, L_k]$$

or, equivalently as

$$\begin{aligned} (\mathrm{ad}_L\, L)_\gamma &= \begin{pmatrix} \gamma_{00}\, \mathrm{ad}_0 & \gamma_{11}\, \mathrm{ad}_1 \\ \gamma_{10}\, \mathrm{ad}_1 & \gamma_{01}\, \mathrm{ad}_0 \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{00}\, \mathrm{ad}_0\, L_0 + \gamma_{11}\, \mathrm{ad}_1\, L_1 \\ \gamma_{10}\, \mathrm{ad}_1\, L_0 + \gamma_{01}\, \mathrm{ad}_0\, L_1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{00}[L_0, L_0] + \gamma_{11}[L_1, L_1] \\ \gamma_{01}[L_0, L_1] \end{pmatrix}. \end{aligned} \tag{3.10}$$

Here $\gamma_{10}\, \mathrm{ad}_1\, L_0$ and $\gamma_{01}\, \mathrm{ad}_0\, L_1$ are the same due to (2.7) and (3.8).

In the generic case, $\kappa = (1)$, we have the three $\mathbb{Z}_2$-graded contractions of L^κ given by (3.5):

$$\text{I.} \qquad 0 \neq [L_0, L_0]_\varepsilon \subseteq L_0 \qquad 0 \neq [L_0, L_1]_\varepsilon \subseteq L_1 \qquad [L_1, L_1]_\varepsilon = 0 \tag{3.11}$$

$$\text{II.} \qquad [L_0, L_0]_\varepsilon = [L_0, L_1]_\varepsilon = 0 \qquad 0 \neq [L_1, L_1]_\varepsilon \subseteq L_0 \tag{3.12}$$

$$\text{III.} \qquad 0 \neq [L_0, L_0]_\varepsilon \subseteq L_0 \qquad [L_0, L_1]_\varepsilon = [L_1, L_1]_\varepsilon = 0. \tag{3.13}$$

Next consider the non-generic cases. Let

$$\kappa = \begin{pmatrix} \cdot & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that such a grading structure is not a result of a contraction of the generic one. In this case the set (2.17) imposes no restriction on γ. According to (2.9), we find the contractions given by

$$\begin{aligned} \varepsilon &= \gamma^{\mathrm{I}} \bullet \kappa = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \\ \varepsilon &= \gamma^{\mathrm{II}} \bullet \kappa = \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix} \\ \varepsilon &= \gamma^{\mathrm{III}} \bullet \kappa = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}. \end{aligned} \tag{3.14}$$

Thus there are now two non-trivial continuous contractions:

$$[L_0, L_0]_\varepsilon = [L_1, L_1]_\varepsilon = 0 \qquad 0 \neq [L_0, L_1]_\varepsilon \subseteq L_1 \tag{3.15}$$

and the one given in (3.12).

It remains to take up $\kappa = \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$ with the contractions determined by

$$\begin{aligned} \varepsilon &= \gamma^{\mathrm{I}} \bullet \kappa = \gamma^{\mathrm{III}} \bullet \kappa = \begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix} \\ \varepsilon &= \gamma^{\mathrm{II}} \bullet \kappa = \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix}. \end{aligned} \tag{3.16}$$

Hence there are the two non-trivial contractions given by (3.12) and (3.13).

4. Contractions of $\mathbb{Z}_3$-graded Lie algebras and superalgebras

In this section the graded group is $\mathbb{Z}_3$, the cyclic group of three elements. We consider any Lie algebra or superalgebra L which admits a $\mathbb{Z}_3$-grading. Hence we have

$$L = L_0 \oplus L_1 \oplus L_2 \tag{4.1}$$

and in the generic case, $\kappa = (1)$, also

$$0 \neq [L_j, L_k] \subseteq L_{j+k} \qquad j, k, j+k (\mathrm{mod}\ 3). \tag{4.2}$$

Then (2.17) can be rewritten as

$$\gamma_{00}\gamma_{0m} = \gamma_{0m}^2 \tag{4.3a}$$

$$\gamma_{01}\gamma_{1m} = \gamma_{1m}\gamma_{0,m+1} = \gamma_{0m}\gamma_{1m} \tag{4.3b}$$

$$\gamma_{02}\gamma_{2m} = \gamma_{2m}\gamma_{0,m+2} = \gamma_{0m}\gamma_{2m} \tag{4.3c}$$

$$\gamma_{11}\gamma_{2m} = \gamma_{1m}\gamma_{1,m+1} \tag{4.3d}$$

$$\gamma_{22}\gamma_{1m} = \gamma_{2m}\gamma_{2,m+2} \tag{4.3e}$$

$$\gamma_{12}\gamma_{0m} = \gamma_{2m}\gamma_{1,m+2} = \gamma_{1m}\gamma_{2,m+1} \qquad m = 0, 1, 2. \tag{4.3f}$$

Occasionally it is more convenient to rewrite (4.3) in the following equivalent form.

$$\begin{aligned}
&\gamma_{0j}(\gamma_{00}-\gamma_{0j})=0\\
&\gamma_{12}(\gamma_{00}-\gamma_{0j})=0\\
&\gamma_{jj}(\gamma_{01}-\gamma_{02})=0\\
&\gamma_{12}(\gamma_{01}-\gamma_{02})=0\\
&\gamma_{11}\gamma_{22}=\gamma_{12}\gamma_{0j} \qquad j=1,2.
\end{aligned} \tag{4.3'}$$

In particular, it is easier to decide whether a contraction is discrete from (4.3′) rather than from (4.3).

In order to find non-trivial solutions of (4.3), let us proceed as follows.

4.1.

First consider the solution with $\gamma_{00}=1$. (If one would have $0\neq\gamma_{00}\neq 1$, the renormalization (2.10) of L_0 would allow one to change γ_{00} to 1.) Then from (4.3*a*) we have

$$\gamma_{01} \text{ and } \gamma_{02}=1 \text{ or } 0. \tag{4.4}$$

Hence we need to consider one-by-one the three cases

$$\begin{aligned}
&\text{(a)} \qquad \gamma_{01}=\gamma_{02}=1\\
&\text{(b)} \qquad \gamma_{01}=\gamma_{02}=0\\
&\text{(c)} \qquad \gamma_{01}=1,\gamma_{02}=0 \qquad \text{and} \qquad \gamma_{01}=0,\gamma_{02}=1.
\end{aligned} \tag{4.5}$$

Case (a). Using $\gamma_{00}=\gamma_{01}=\gamma_{02}=1$ in (4.3), we get

$$\begin{aligned}
&\gamma_{11}\gamma_{2m}=\gamma_{1m}\gamma_{1,m+1}\\
&\gamma_{22}\gamma_{1m}=\gamma_{2m}\gamma_{2,m+2}\\
&\gamma_{12}=\gamma_{2m}\gamma_{1,m+2}=\gamma_{1m}\gamma_{2,m+1} \qquad m=0,1,2.
\end{aligned}$$

These equations simplify to just one:

$$\gamma_{12}=\gamma_{11}\gamma_{22}.$$

Hence we get the non-trivial solutions

$$\gamma^{\rm I}=\begin{pmatrix}1&1&1\\1&1&\cdot\\1&\cdot&\cdot\end{pmatrix} \qquad \gamma^{\rm II}=\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\1&\cdot&1\end{pmatrix} \qquad \gamma^{\rm III}=\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}. \tag{4.6}$$

Case (b). Using $\gamma_{00}=1,\gamma_{01}=\gamma_{02}=0$ in (4.3), we get after obvious simplifications

$$0=\gamma_{11}\gamma_{22} \qquad \gamma_{12}=0. \tag{4.7}$$

Hence the non-trivial solutions are

$$\gamma^{\mathrm{IV}} = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad \gamma^{\mathrm{V}} = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \qquad \gamma^{\mathrm{VI}} = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}. \tag{4.8}$$

Case (c). Using $\gamma_{00} = \gamma_{01} = 1, \gamma_{02} = 0$ in (4.3), yields only

$$\gamma^{\mathrm{VII}} = \begin{pmatrix} 1 & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}. \tag{4.9}$$

Interchanging γ_{01} and γ_{02}, we get as well

$$\gamma^{\mathrm{VIII}} = \begin{pmatrix} 1 & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}. \tag{4.10}$$

4.2.

Next we consider the solutions with $\gamma_{00} = 0$. From (4.3*a*) we have $\gamma_{01} = \gamma_{02} = 0$. Consequently (4.3) becomes

$$\gamma_{11}\gamma_{22} = 0 \qquad \gamma_{12} = a \in \mathbb{C}. \tag{4.11}$$

However, $a \neq 0$ can be transformed to 1 by (2.10). We get

$$\begin{gathered} \gamma^{\mathrm{IX}} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & 1 \\ \cdot & 1 & \cdot \end{pmatrix} \qquad \gamma^{\mathrm{X}} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & 1 \end{pmatrix} \\ \gamma^{\mathrm{XI}} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix} \qquad \gamma^{\mathrm{XII}} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \\ \gamma^{\mathrm{XIII}} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}. \end{gathered} \tag{4.12}$$

In many important special cases the subspaces L_1 and L_2 are isomorphic and can be relabelled as L_{-1} and L_{-2}. Then the pairs of solution

$$\begin{gathered} \gamma^{\mathrm{I}} \text{ and } \gamma^{\mathrm{II}} \qquad \gamma^{\mathrm{IV}} \text{ and } \gamma^{\mathrm{V}} \qquad \gamma^{\mathrm{VII}} \text{ and } \gamma^{\mathrm{VIII}} \\ \gamma^{\mathrm{IX}} \text{ and } \gamma^{\mathrm{X}} \qquad \gamma^{\mathrm{XII}} \text{ and } \gamma^{\mathrm{XIII}} \end{gathered} \tag{4.13}$$

give equivalent contractions of L.

Note that the sets of solutions

$$\begin{gathered} \{\gamma^{\mathrm{I}}, \ldots, \gamma^{\mathrm{XIII}}, (0)\} \\ \{\gamma^{\mathrm{I}}, \gamma^{\mathrm{III}}, \gamma^{\mathrm{IV}}, \gamma^{\mathrm{VI}}, \gamma^{\mathrm{VII}}, \gamma^{\mathrm{IX}}, \gamma^{\mathrm{XI}}, \gamma^{\mathrm{XII}}, (0)\} \\ \{\gamma^{\mathrm{II}}, \gamma^{\mathrm{III}}, \gamma^{\mathrm{V}}, \gamma^{\mathrm{VI}}, \gamma^{\mathrm{VIII}}, \gamma^{\mathrm{X}}, \gamma^{\mathrm{XI}}, \gamma^{\mathrm{XIII}}, (0)\} \end{gathered} \tag{4.14}$$

are closed under the composition rule (2.8). Here (0) is the trivial solution. The solutions $\gamma^{\mathrm{I}}, \gamma^{\mathrm{II}}, \gamma^{\mathrm{III}}, \gamma^{\mathrm{IX}}$–$\gamma^{\mathrm{XIII}}$ are continuous while γ^{IV}–γ^{VIII} are discrete.

Occasionally it is convenient to write the adjoint action of a $\mathbb{Z}_3$-graded Lie algebra L in a way analogous to (3.10):

$$(\mathrm{ad}_L\, L)_\gamma = \begin{pmatrix} \gamma_{00}\,\mathrm{ad}_0 & \gamma_{21}\,\mathrm{ad}_2 & \gamma_{12}\,\mathrm{ad}_1 \\ \gamma_{10}\,\mathrm{ad}_1 & \gamma_{01}\,\mathrm{ad}_0 & \gamma_{22}\,\mathrm{ad}_2 \\ \gamma_{20}\,\mathrm{ad}_2 & \gamma_{11}\,\mathrm{ad}_1 & \gamma_{02}\,\mathrm{ad}_0 \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} \gamma_{00}[L_0, L_0] + \gamma_{12}[L_1, L_2] \\ \gamma_{10}[L_0, L_1] + \gamma_{22}[L_2, L_2] \\ \gamma_{20}[L_0, L_2] + \gamma_{11}[L_1, L_1] \end{pmatrix}. \tag{4.15}$$

Let us now consider an example of Lie algebras L^κ with non-generic κ, say

$$\kappa = \begin{pmatrix} \cdot & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{4.16}$$

All its contractions are found as

$$\varepsilon = \gamma^{\mathrm{A}} \bullet \kappa \qquad \text{for A} = \mathrm{I}, \ldots, \mathrm{XIII}. \tag{4.17}$$

Thus one finds the following continuous contractions

$$\begin{pmatrix} \cdot & 1 & 1 \\ 1 & 1 & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & 1 & 1 \\ 1 & \cdot & \cdot \\ 1 & \cdot & 1 \end{pmatrix} \qquad \begin{pmatrix} \cdot & 1 & 1 \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \qquad \gamma^{\mathrm{IX}}, \ldots, \gamma^{\mathrm{XIII}} \tag{4.18}$$

and the discrete ones

$$\begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & 1 & p \\ 1 & \cdot & \cdot \\ p & \cdot & \cdot \end{pmatrix} \qquad 0 \neq p \neq 1. \tag{4.19}$$

The parameter p gives a continuum of contracted Lie algebras which, for different values of p, are not isomorphic. In this case the equalities involving ε_{00} are *absent* from (4.3). As a result the system is solved with arbitrary ε_{01} and ε_{02}. Only one of the two can be transformed to 1 by renormalization of L_0, unless of course $\varepsilon_{01} = \varepsilon_{02} \neq 0$, in which case one gets one of the continuous contractions (4.18).

Another example of a non-generic $\mathbb{Z}_3$-grading pertinent to contractions of $\mathrm{sl}(2, \mathbb{C})$ considered in section 7, is given by

$$\kappa = \begin{pmatrix} \cdot & 1 & 1 \\ 1 & \cdot & 1 \\ 1 & 1 & \cdot \end{pmatrix}. \tag{4.20}$$

In this case the continuous contractions are given by

$$\varepsilon = \kappa \bullet \gamma^{\mathrm{I}} = \begin{pmatrix} \cdot & 1 & 1 \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \qquad \varepsilon = \gamma^{\mathrm{XI}} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix} \tag{4.21}$$

and the discrete ones by

$$\varepsilon = \kappa \bullet \gamma^{\mathrm{VIII}} = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & 1 & p \\ 1 & \cdot & \cdot \\ p & \cdot & \cdot \end{pmatrix} \qquad p \neq 1. \tag{4.22}$$

Here the last matrix is also obtained from $\kappa \bullet \gamma^{\mathrm{I}}$, followed by the investigation of the admissible range of values of non-zero matrix elements as in the previous example.

5. Contractions of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and superalgebras

The grading group in this section is $\mathbb{Z}_2 \times \mathbb{Z}_2$ the tensor product of two cyclic groups of order 2. We consider any Lie algebra or superalgebra which admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading.

We have

$$L = L_{00} \oplus L_{01} \oplus L_{10} \oplus L_{11} \tag{5.1}$$

and also

$$[L_{pq}, L_{rs}] \subseteq L_{p+r,q+s} \tag{5.2}$$

where the subscripts have two components, each read modulo 2. As before we first consider the generic case where none of the commutators (5.2) is identically zero.

In order to simplify our notations we introduce letter-symbols for the two component subscripts:

$$a = 00 \qquad b = 01 \qquad c = 10 \qquad d = 11. \tag{5.3}$$

Consequently we have

$$\begin{aligned} &a + a = a \qquad a + b = b \qquad a + c = c \qquad a + d = d \\ &2a = 2b = 2c = 2d = a \\ &b + c = d \qquad b + d = c \qquad c + d = b. \end{aligned} \tag{5.4}$$

The matrices $\boldsymbol{\kappa}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\varepsilon}$ now have the form

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{00,00} & \varepsilon_{00,01} & \varepsilon_{00,10} & \varepsilon_{00,11} \\ \varepsilon_{01,00} & \varepsilon_{01,01} & \varepsilon_{01,10} & \varepsilon_{01,11} \\ \varepsilon_{10,00} & \varepsilon_{10,01} & \varepsilon_{10,10} & \varepsilon_{10,11} \\ \varepsilon_{11,00} & \varepsilon_{11,01} & \varepsilon_{11,10} & \varepsilon_{11,11} \end{pmatrix} = \begin{pmatrix} \varepsilon_{aa} & \varepsilon_{ab} & \varepsilon_{ac} & \varepsilon_{ad} \\ \varepsilon_{ba} & \varepsilon_{bb} & \varepsilon_{bc} & \varepsilon_{bd} \\ \varepsilon_{ca} & \varepsilon_{cb} & \varepsilon_{cc} & \varepsilon_{cd} \\ \varepsilon_{da} & \varepsilon_{db} & \varepsilon_{dc} & \varepsilon_{dd} \cdot \end{pmatrix}. \tag{5.5}$$

The equations (2.17) can now be rewritten as

$$\begin{aligned} &\gamma_{ax}(\gamma_{aa} - \gamma_{ax}) = 0 \\ &\gamma_{xx}(\gamma_{aa} - \gamma_{ax}) = 0 \\ &\gamma_{xy}(\gamma_{ax} - \gamma_{az}) = \gamma_{xy}(\gamma_{ax} - \gamma_{ay}) = 0 \\ &\gamma_{xx}\gamma_{ay} = \gamma_{xy}\gamma_{xz} \\ &\gamma_{xx}\gamma_{yz} = \gamma_{yy}\gamma_{zx} \end{aligned} \tag{5.6}$$

where a is as in (5.4) and distinct letters x, y, z denote distinct values from the set b, c, d of (5.4). Rather than trying to solve (5.6) directly, one may first observe that the system (5.6) is solved by tensor products of $\mathbb{Z}_2$-solutions (for the generic case). We have found three non-trivial $\mathbb{Z}_2$-solutions: $\gamma^{\mathrm{I}}, \gamma^{\mathrm{II}}, \gamma^{\mathrm{III}}$ of (3.5). Denoting by γ^0 the first of the trivial solutions of (3.4), we can write down immediately the 16 following solutions of (5.6):

$$\gamma^{\mathrm{AB}} = \gamma^{\mathrm{A}} \otimes \gamma^{\mathrm{B}} \qquad \mathrm{A, B} \in \{0, \mathrm{I}, \mathrm{II}, \mathrm{III}\}$$

Table 1. Solutions of (5.6) providing $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded contractions for the generic case as tensor products of $\mathbb{Z}_2$-graded contractions given by γ^0, γ^{I}, γ^{II} and γ^{III}. The subscript d denotes a discrete contraction; all other contractions are continuous.

$\otimes$	γ^0	γ^{I}	γ^{II}	γ^{III}
γ^0	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$
γ^{I}	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$
γ^{II}	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
γ^{III}	$\begin{pmatrix} 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$

where only $\gamma^0 \otimes \gamma^0$ is trivial. The solutions are shown in table 1. However, the system (5.6) also admits other solutions. By a direct computation one finds the contraction matrices given in table 2. Altogether there are 39 non-trivial solutions of (5.6).

The graded commutation relation in the matrix representation analogous to (3.9) becomes

$$\begin{pmatrix} \gamma_{aa}\,\mathrm{ad}_a & \gamma_{bb}\,\mathrm{ad}_b & \gamma_{cc}\,\mathrm{ad}_c & \gamma_{dd}\,\mathrm{ad}_d \\ \gamma_{ba}\,\mathrm{ad}_b & \gamma_{ab}\,\mathrm{ad}_a & \gamma_{dc}\,\mathrm{ad}_d & \gamma_{cd}\,\mathrm{ad}_c \\ \gamma_{ca}\,\mathrm{ad}_c & \gamma_{db}\,\mathrm{ad}_d & \gamma_{ac}\,\mathrm{ad}_a & \gamma_{bd}\,\mathrm{ad}_b \\ \gamma_{da}\,\mathrm{ad}_d & \gamma_{cb}\,\mathrm{ad}_c & \gamma_{bc}\,\mathrm{ad}_b & \gamma_{ad}\,\mathrm{ad}_a \end{pmatrix} \begin{pmatrix} L_a \\ L_b \\ L_c \\ L_d \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{aa}[L_a, L_a] + \gamma_{bb}[L_b, L_b] + \gamma_{cc}[L_c, L_c] + \gamma_{dd}[L_d, L_d] \\ \gamma_{ab}[L_a, L_b] + \gamma_{cd}[L_c, L_d] \\ \gamma_{ac}[L_a, L_c] + \gamma_{bd}[L_b, L_d] \\ \gamma_{ad}[L_a, L_d] + \gamma_{bc}[L_b, L_c] \end{pmatrix}. \tag{5.7}$$

The contractions for Lie algebras with non-generic grading structure, i.e.

$$\boldsymbol{\kappa} \neq \gamma^0 \otimes \gamma^0 = (1) \tag{5.8}$$

are again found using (2.8). For example, let

$$\boldsymbol{\kappa} = \begin{pmatrix} \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{5.9}$$

then combining it with the entries in table 1, we get the following non-trivial

Table 2. Solutions of (5.6) providing $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded contractions for the generic case which cannot be given as tensor products of two $\mathbb{Z}_2$-graded contraction matrices. The subscript d indicates a discrete contraction.

$\begin{pmatrix} 1 & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} 1 & 1 & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}_{\mathrm{d}}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}_{\mathrm{d}}$
$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

contraction matrices:

$$\begin{pmatrix} \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$$
$$\begin{pmatrix} \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{5.10}$$
$$\begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and

$$\begin{array}{llll} \gamma^{0} \otimes \gamma^{\mathrm{II}} & \gamma^{\mathrm{I}} \otimes \gamma^{\mathrm{II}} & \gamma^{\mathrm{II}} \otimes \gamma^{0} & \gamma^{\mathrm{II}} \otimes \gamma^{\mathrm{I}} \\ \gamma^{\mathrm{II}} \otimes \gamma^{\mathrm{II}} & \gamma^{\mathrm{II}} \otimes \gamma^{\mathrm{III}} & \gamma^{\mathrm{III}} \otimes \gamma^{\mathrm{II}} & \end{array} \tag{5.11}$$

as they appear in table 1. Moreover, there are 21 others obtained by the composition of (5.9) with the matrices of table 2. Altogether there are 35 distinct non-trivial contractions of (5.9). Some of those solutions depend on parameters. Among them one finds, for example,

$$\begin{pmatrix} \cdot & 1 & p & q \\ 1 & \cdot & \cdot & \cdot \\ p & \cdot & \cdot & \cdot \\ q & \cdot & \cdot & \cdot \end{pmatrix} \qquad p, q \in \mathbb{C}$$

which coincides with one of (5.10) for $p = q = 0$.

In section 8 we will encounter yet another non-generic $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading structure. Namely,

$$\kappa = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 \\ \cdot & 1 & 1 & \cdot \end{pmatrix}. \tag{5.12}$$

Its composition (2.8) with those in tables 1 and 2 yield the following non-trivial contractions:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & 1 & \cdot \end{pmatrix}$$
$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{5.13}$$

6. Examples of contractions with $\mathbb{Z}_2$-grading

In this section we consider examples of specific algebras which admit $\mathbb{Z}_2$-gradings and the related contractions.

Example 1. Consider the simple Lie algebra A_2 as represented by the matrices

$$X \in \mathbb{C}^{3\times3} \qquad \operatorname{tr} X = 0. \tag{6.1}$$

There are two non-equivalent $\mathbb{Z}_2$-gradings of A_2. Both of them lead to the decomposition (3.1). However, they differ in what are the subspaces L_0 and L_1. In one of the cases

$$\begin{aligned} L_0 &= \{X \mid X = -X^{\mathrm{T}}\} \simeq A_1 \\ L_1 &= \{X \mid X = X^{\mathrm{T}}\}. \end{aligned} \tag{6.2}$$

Thus $\dim L_0 = 3$ and $\dim L_1 = 5$. Explicitly L_0 and L_1 can be taken as the matrices

$$L_0 = \begin{pmatrix} \cdot & a & b \\ -a & \cdot & c \\ -b & -c & \cdot \end{pmatrix} \qquad L_1 = \begin{pmatrix} d & e & f \\ e & g & h \\ f & h & -d-g \end{pmatrix} \tag{6.3}$$

where $a, b, c, d, e, f, g, h \in \mathbb{C}$.

The second $\mathbb{Z}_2$-grading of A_2 can be given as

$$L_0 = \begin{pmatrix} a & b & \cdot \\ c & d & \cdot \\ \cdot & \cdot & -a-d \end{pmatrix} \qquad L_1 = \begin{pmatrix} \cdot & \cdot & e \\ \cdot & \cdot & f \\ g & h & \cdot \end{pmatrix} \tag{6.4}$$

where $a, b, \ldots, h \in \mathbb{C}$, with $\dim L_0 = \dim L_1 = 4$. Clearly the two gradings cannot be equivalent (under the action of the group of automorphisms of L) because the dimensions of their grading subspaces differ.

Both $\mathbb{Z}_2$-gradings of A_2 have the generic grading structure, $\kappa = (1)$, since none of the commutators (3.2) is identically zero. Consequently the solutions (3.5) apply to either of them, the contracted commutators being those of (3.11)–(3.13). Using the format of (3.10) we write the contractions in the following form:

$$\gamma^{\mathrm{I}}: \qquad \begin{pmatrix} \mathrm{ad}_0 & 0 \\ \mathrm{ad}_1 & \mathrm{ad}_0 \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 \end{pmatrix} = \begin{pmatrix} [L_0, L_0] \\ [L_0, L_1] \end{pmatrix} \tag{6.5}$$

$$\gamma^{\mathrm{II}}: \qquad \begin{pmatrix} 0 & \mathrm{ad}_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 \end{pmatrix} = \begin{pmatrix} [L_1, L_1] \\ 0 \end{pmatrix} \tag{6.6}$$

$$\gamma^{\mathrm{III}}: \qquad \begin{pmatrix} \mathrm{ad}_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 \end{pmatrix} = \begin{pmatrix} [L_0, L_0] \\ 0 \end{pmatrix}. \tag{6.7}$$

Let us now use the two $\mathbb{Z}_2$-gradings of A_2 in (6.5)–(6.7). First consider (6.3). Since in this case $[L_0, L_0] = L_0$, $[L_1, L_1] = L_0$, and $[L_0, L_1] = L_1$, the derived algebra DL^ϵ is isomorphic to L^ϵ in the case (6.5), $DL^\epsilon = L_0$ for (6.6) and (6.7). Moreover, (6.5) has a non-trivial Levy decomposition into o(3) and a five-dimensional (5D) Abelian ideal, (6.6) is indecomposable nilpotent with 3D centre (degree 2 nilpotency is a consequence of $\mathbb{Z}_2$-grading), and (6.7) is decomposable: o(3) and 5D Abelian algebra commuting with o(3).

In the case (6.4) we have $L_0 \simeq \mathrm{gl}(2,\mathbb{C})$, $L_0' \simeq \mathrm{sl}(2,\mathbb{C})$,

$$\begin{aligned} &[L_0, L_0] = \begin{pmatrix} a' & b' & \cdot \\ c' & -a' & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = L_0' \subset L_0 \\ &[L_0, L_1] = L_1 \qquad [L_1, L_1] = L_0. \end{aligned} \tag{6.8}$$

Hence for (6.5), DL^ϵ is not isomorphic to L^ϵ because $\dim DL^\epsilon = 7$; equation (6.6) gives $\dim DL^\epsilon = 4$; for (6.7) we have $DL^\epsilon \simeq \mathrm{sl}(2,\mathbb{C})$. Furthermore, (6.5) decomposes into (7+1)D subalgebras, (6.6) is indecomposable nilpotent with 4D centre, and (6.7) is decomposable $(3 + 1 + \cdots + 1)$, containing $\mathrm{sl}(2,\mathbb{C})$.

Example 2. Consider the affine Kac–Moody algebra $A_1^{(1)}$ spanned by the generators

$$\begin{aligned} &\begin{pmatrix} t^k & \cdot \\ \cdot & -t^k \end{pmatrix} \qquad \begin{pmatrix} \cdot & t^k \\ \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot \\ t^k & \cdot \end{pmatrix} \\ &\not{t} = \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix} \qquad -\infty < k < \infty \end{aligned} \tag{6.9}$$

and equipped with the modified matrix commutation rules, the modification occurring only in

$$\left[\begin{pmatrix} \cdot & t^k \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^j & \cdot \end{pmatrix} \right] = \begin{pmatrix} t^{k+j} & \cdot \\ \cdot & -t^{k+j} \end{pmatrix} + k\delta_{k+j,0}\,\not{t} \tag{6.10}$$

when $k + j = 0$.

A $\mathbb{Z}_2$-grading of $A_1^{(1)}$ with generic grading structure, $\kappa = (1)$ is given by the following generators of grading subspaces:

$$L_0 = \left\{ \begin{pmatrix} t^{2k} & \cdot \\ \cdot & -t^{2k} \end{pmatrix}, \begin{pmatrix} \cdot & t^{2k} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{2k} & \cdot \end{pmatrix}, \not{c} \right\}$$
$$L_1 = \left\{ \begin{pmatrix} t^{2k+1} & \cdot \\ \cdot & -t^{2k+1} \end{pmatrix}, \begin{pmatrix} \cdot & t^{2k+1} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{2k+1} & \cdot \end{pmatrix} \right\}. \tag{6.11}$$

Here L_0 is isomorphic to the whole algebra $A_1^{(1)}$, and L_1 is isomorphic to the adjoint representation $\operatorname{ad} A_1^{(1)}$ (centre mapped to zero).

The contractions (6.5)–(6.7) give respectively $A_1^{(1)} \ltimes \operatorname{ad} A_1^{(1)}$, infinite-dimensional nilpotent Lie algebra (degree 2 nilpotency), and a decomposable algebra consisting of $A_1^{(1)}$ and an infinite-dimensional Abelian Lie algebra commuting with $A_1^{(1)}$.

The algebra $A_1^{(1)}$ admits another $\mathbb{Z}_2$-grading generated as follows:

$$L_0 = \left\{ \begin{pmatrix} t^{k} & \cdot \\ \cdot & -t^{k} \end{pmatrix}, \not{c} \right\}$$
$$L_1 = \left\{ \begin{pmatrix} \cdot & t^{k} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{k} & \cdot \end{pmatrix} \right\} \qquad -\infty < k < \infty \tag{6.12}$$

with the grading structure

$$\kappa = \begin{pmatrix} \cdot & 1 \\ 1 & 1 \end{pmatrix}.$$

In section 3 we found the two non-trivial contractions in this case, namely (3.12) and (3.15). The first is nilpotent, the second one is solvable.

For completeness let us also point out the following $\mathbb{Z}_2$-grading of $A_1^{(1)}$ which is clearly not equivalent to any other:

$$L_0 = \left\{ \begin{pmatrix} t^{2k} & \cdot \\ \cdot & -t^{2k} \end{pmatrix}, \begin{pmatrix} \cdot & t^{2k+1} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{2k+1} & \cdot \end{pmatrix}, \not{c} \right\}$$
$$L_1 = \left\{ \begin{pmatrix} t^{2k+1} & \cdot \\ \cdot & -t^{2k+1} \end{pmatrix}, \begin{pmatrix} \cdot & t^{2k} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{2k} & \cdot \end{pmatrix} \right\}. \tag{6.13}$$

Example 3. Superalgebras carry by definition a natural $\mathbb{Z}_2$-grading. Let us consider the superalgebra of 3×3 supermatrices of the following form. The even part L_0 is the same as in (6.4), and the odd part L_1 is isomorphic to that in (6.4) except that, being a superalgebra, one has now to equip it with supercommutation rules:

$$[L_0, L_0] = L_0' \subset L_0 \qquad [L_0, L_1] = L_1 \qquad \{L_1, L_1\} = L_0. \tag{6.14}$$

The contractions are again described by (6.5)–(6.7), the discussion following (6.8) applies, remembering that $\{L_1, L_1\}$ is an anticommutator.

Somewhat more amusing is the $\mathbb{Z}_2$-grading of this superalgebra where L_0 and L_1 both have non-trivial even and odd parts:

$$L_0 = \left\{ \begin{pmatrix} a & \cdot & \cdot \\ \cdot & b & \cdot \\ \cdot & \cdot & -a-b \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & A \\ \cdot & \cdot & B \\ B & A & \cdot \end{pmatrix} \right\}$$
$$L_1 = \left\{ \begin{pmatrix} \cdot & c & \cdot \\ d & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & C \\ \cdot & \cdot & D \\ -D & -C & \cdot \end{pmatrix} \right\}. \tag{6.15}$$

Here $a, b, c, d \in \mathbb{C}$ are the even variables and A, B, C, D are the odd ones. L_0 is now a superalgebra with 2D even part and 2D odd part; L_1 is a 4D representation space of L_0 which also decomposes $(2+2)$ into even and odd subspaces. The grading is generic, therefore the contractions are again described by (6.5)-(6.7).

Example 4. Finally, let us consider the simple Lie algebra A_1. It has one $\mathbb{Z}_2$-grading. We obtain it by putting $t = 1$ in (6.12) and disregarding the centre generator $\not{t}$:

$$L_0 = \left\{ \begin{pmatrix} h & \cdot \\ \cdot & -h \end{pmatrix} \right\} \qquad L_1 = \left\{ \begin{pmatrix} \cdot & e \\ f & \cdot \end{pmatrix} \right\} \qquad h, e, f \in \mathbb{C}. \tag{6.16}$$

The grading structure is $\kappa = \begin{pmatrix} \cdot & 1 \\ 1 & 1 \end{pmatrix}$ with two continuous contractions given by (3.12) and (3.15). The first of them is a 3D indecomposable nilpotent Lie algebra (Heisenberg algebra), the second one is an indecomposable solvable Lie algebra of transformations of a 2D complex plane L_1.

7. Examples of contractions with $\mathbb{Z}_3$-grading

Example 1. The simple Lie algebra of traceless matrices $\mathbb{C}^{3\times 3}$ has two non-equivalent $\mathbb{Z}_3$-gradings. The grading decompositions consist of the subspaces L_0, L_1, L_2 generated in the following way:

$$\begin{aligned}
L_0 &= \left\{ \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \right\} \\
L_1 &= \left\{ \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \right\} \\
L_2 &= \left\{ \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\}
\end{aligned} \tag{7.1}$$

and

$$\begin{aligned}
L_0 &= \left\{ \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -2 \end{pmatrix}, \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\} \\
L_1 &= \left\{ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\} \\
L_2 &= \left\{ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \right\}.
\end{aligned} \tag{7.2}$$

It is straightforward to verify the grading property (2.2) of the commutators in these cases.

The grading structures κ are respectively given by (4.16) and by

$$\kappa = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \cdot & 1 \\ 1 & 1 & \cdot \end{pmatrix}. \tag{7.3}$$

In the first case the possible contractions are those of (4.18) and (4.19). Contractions of the case (7.2) are among those studied in [8].

Consider the last of the matrices (4.19). The contracted Lie algebra is solvable and indecomposable. Its non-zero commutators are

$$[L_0, L_1] = L_1 \qquad [L_0, L_2] = pL_2 \qquad p \neq 0 \tag{7.4}$$

the parameter p could be transformed to 1 by the normalization $L_0 \longrightarrow L_0' = p^{-1}L_0$. However, in that case the first commutator would take the form $[L_0', L_1] = p^{-1}L_1$, i.e. the parameter would appear there.

For the grading (7.2) the contractions are found using κ of (7.3) composed with γ^{I}–γ^{XIII} of section 4.

Example 2. The affine algebra of $A_1^{(1)}$ has three non-equivalent $\mathbb{Z}_3$-gradings.

$$L_0 = \{A_1 \otimes t^{3k}, \not{c}\} \qquad L_1 = \{A_1 \otimes t^{3k+1}\} \qquad L_2 = \{A_1 \otimes t^{3k+2}\} \tag{7.5}$$

where $-\infty < k < \infty$ and A_1 denote the 3D simple Lie algebra.

$$L_0 = \left\{ \begin{pmatrix} t^k & \cdot \\ \cdot & -t^k \end{pmatrix}, \not{c} \right\} \qquad L_1 = \left\{ \begin{pmatrix} \cdot & t^k \\ \cdot & \cdot \end{pmatrix} \right\} \qquad L_2 = \left\{ \begin{pmatrix} \cdot & \cdot \\ t^k & \cdot \end{pmatrix} \right\} \tag{7.6}$$

which is clearly the refined $\mathbb{Z}_2$-grading of (6.12), the subspace L_1 there being split into L_1 and L_2 in (7.6). The algebra $A_1^{(1)}$ admits another $\mathbb{Z}_3$-grading generated as follows:

$$\begin{aligned} L_0 &= \left\{ \begin{pmatrix} t^{3k} & \cdot \\ \cdot & -t^{3k} \end{pmatrix}, \begin{pmatrix} \cdot & t^{3k+2} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{3k+1} & \cdot \end{pmatrix}, \not{c} \right\} \\ L_1 &= \left\{ \begin{pmatrix} \cdot & t^{3k} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{3k+2} & \cdot \end{pmatrix}, \begin{pmatrix} t^{3k+1} & \cdot \\ \cdot & -t^{3k+1} \end{pmatrix} \right\} \\ L_2 &= \left\{ \begin{pmatrix} \cdot & \cdot \\ t^{3k} & \cdot \end{pmatrix}, \begin{pmatrix} t^{3k+2} & \cdot \\ \cdot & -t^{3k+2} \end{pmatrix}, \begin{pmatrix} \cdot & t^{3k+1} \\ \cdot & \cdot \end{pmatrix} \right\}. \end{aligned} \tag{7.7}$$

The gradings (7.5) and (7.7) are clearly generic, hence the corresponding contractions are γ^{I}–γ^{XIII} of section 4. For (7.6) we have

$$\kappa = \begin{pmatrix} \cdot & 1 & 1 \\ 1 & \cdot & 1 \\ 1 & 1 & \cdot \end{pmatrix} \tag{7.8}$$

with the contractions given in (4.21) and (4.22).

Example 3. There is only one $\mathbb{Z}_3$-grading of osp(2, 1), up to equivalence. Namely,

$$\begin{aligned}
L_0 &= \left\{ \begin{pmatrix} a & \cdot & \cdot \\ \cdot & -a & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} b & \cdot & \cdot \\ \cdot & b & \cdot \\ \cdot & \cdot & -2b \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & X \\ \cdot & \cdot & \cdot \\ \cdot & X & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & Y \\ Y & \cdot & \cdot \end{pmatrix} \right\} \\
L_1 &= \left\{ \begin{pmatrix} \cdot & c & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & V \\ -V & \cdot & \cdot \end{pmatrix} \right\} \\
L_2 &= \left\{ \begin{pmatrix} \cdot & \cdot & \cdot \\ e & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & W \\ \cdot & \cdot & \cdot \\ \cdot & -W & \cdot \end{pmatrix} \right\}
\end{aligned} \tag{7.9}$$

which is an obvious refinement of the $\mathbb{Z}_2$-grading (6.15) by splitting its subspace L_1 into two. Direct verification confirms that (7.9) is generic. Due to the finer grading, the possible contractions are more varied than for (6.15). They are descibed by γ^{I}–γ^{XIII} of section 4.

Example 4. The $\mathbb{Z}_3$-grading of A_1 is obtained by putting $t = 1$ in (7.6) and ignoring $\not{t}$ which now commutes with A_1;

$$L_0 = \left\{ \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix} \right\} \qquad L_1 = \left\{ \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} \right\} \qquad L_2 = \left\{ \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \right\}. \tag{7.10}$$

The grading is an obvious refinement of (6.16). The grading structure is that of (4.20) and the discrete and continuous contractions are determined respectively by (4.21) and (4.22). While the continuous contractions yield the same Lie algebras as in the $\mathbb{Z}_2$-graded case, the discrete contractions (4.22) have no analogue in the $\mathbb{Z}_2$-case. They give either a decomposable $2+1$ solvable algebra, or an indecomposable $(0 \neq p \neq 1)$ one of non-uniform scaling transformations in the L_1 plane. The latter can be described using (7.10) and (4.15) as follows

$$\begin{pmatrix} 0 & 0 & 0 \\ \mathrm{ad}_1 & \mathrm{ad}_0 & 0 \\ p\,\mathrm{ad}_2 & 0 & p\,\mathrm{ad}_0 \end{pmatrix} \begin{pmatrix} H \\ E \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ [E', H] + [H', E] \\ p[F', H] + p[H', F] \end{pmatrix}. \tag{7.11}$$

Here we have used

$$H = h \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix} \qquad E = e \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} \qquad F = f \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$$

where $h, e, f, h', e', f' \in \mathbb{C}$.

8. Examples of contractions with $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings

In this section we consider the same algebras as in previous examples. In order to get the desired grading we refine the $\mathbb{Z}_2$-gradings of section 6.

Example 1. The $\mathbb{Z}_2$-gradings (6.3) and (6.4) can be combined into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of the simple Lie algebra A_2 of 3×3 traceless matrices. The decomposition (5.1) consists of the four subspaces generated as follows:

$$
\begin{aligned}
L_{00} &= \left\{ \begin{pmatrix} \cdot & 1 & \cdot \\ -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\} \\
L_{10} &= \left\{ \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -2 \end{pmatrix}, \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\} \\
L_{01} &= \left\{ \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & -1 & \cdot \end{pmatrix} \right\} \\
L_{11} &= \left\{ \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix} \right\}.
\end{aligned}
\tag{8.1}
$$

This is a grading with κ given in (5.9) and all contractions determined by composing κ with the matrices of tables 1 and 2. Combining κ, say with $\gamma^{\mathrm{I}} \otimes \gamma^{\mathrm{I}}$ of table 1, we get

$$
\kappa \bullet \gamma^{\mathrm{I}} \otimes \gamma^{\mathrm{I}} = \begin{pmatrix} \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}. \tag{8.2}
$$

The pertinent equations in this case are obtained from (5.6) by removing the equations which contain γ_{aa}. The remaining system of equations is solved with any $\gamma_{ad} \neq 0$ and $\gamma_{ab} = \gamma_{ac} \neq 0$. Therefore the result is an indecomposable solvable Lie algebra with the following non-zero commutators of the grading subspaces

$$
\begin{aligned}
&[L_{00}, L_{01}]_\varepsilon = L_{01} \qquad [L_{00}, L_{10}]_\varepsilon = L_{10} \\
&[L_{00}, L_{11}]_\varepsilon = pL_{11} \qquad [L_{01}, L_{10}]_\varepsilon = L_{11} \qquad p \neq 0.
\end{aligned}
\tag{8.3}
$$

Example 2. Similarly we can combine the $\mathbb{Z}_2$-gradings (6.11) and (6.12), or to refine (6.13) into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of $A_1^{(1)}$:

$$
\begin{aligned}
L_{00} &= \left\{ \begin{pmatrix} t^{2k} & \cdot \\ \cdot & -t^{2k} \end{pmatrix}, \not{t} \right\} \qquad & L_{10} &= \left\{ \begin{pmatrix} \cdot & t^{2k} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{2k} & \cdot \end{pmatrix} \right\} \\
L_{01} &= \left\{ \begin{pmatrix} t^{2k+1} & \cdot \\ \cdot & -t^{2k+1} \end{pmatrix} \right\} \qquad & L_{11} &= \left\{ \begin{pmatrix} \cdot & t^{2k+1} \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ t^{2k+1} & \cdot \end{pmatrix} \right\}
\end{aligned}
\tag{8.4}
$$

with

$$
\kappa = \begin{pmatrix} \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{8.5}
$$

Combining that with, for example $\gamma^{\mathrm{II}} \otimes \gamma^0$ of table 1, we get

$$\varepsilon = \kappa \bullet \gamma^{\mathrm{II}} \otimes \gamma^0 = \begin{pmatrix} \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \bullet \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \end{pmatrix} \tag{8.6}$$

corresponding to an indecomposable infinite-dimensional Lie algebra, with non-zero commutators

$$[L_{10}, L_{10}]_\varepsilon = L_{00} \qquad [L_{10}, L_{11}]_\varepsilon = L_{01} \qquad [L_{11}, L_{11}]_\varepsilon = L_{00}. \tag{8.7}$$

The algebra $A_1^{(1)}$ admits a different $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading:

$$\begin{aligned} &L_{00} = \{\emptyset\} \qquad L_{01} = \left\{ \begin{pmatrix} t^k & \cdot \\ \cdot & -t^k \end{pmatrix} \right\} \qquad L_{10} = \left\{ \begin{pmatrix} \cdot & t^k \\ t^k & \cdot \end{pmatrix} \right\} \\ &L_{11} = \left\{ \begin{pmatrix} \cdot & t^k \\ -t^k & \cdot \end{pmatrix} \right\} \qquad -\infty < k < \infty \end{aligned} \tag{8.8}$$

with κ as in (5.12). In this case there are six non-trivial contractions specified by the matrices ε in (5.13).

Example 3. In an analogous way we can combine the two $\mathbb{Z}_2$-gradings of the superalgebra osp(2, 1) of (6.14) and (6.15) into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading:

$$\begin{aligned} L_{00} &= \left\{ \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -2 \end{pmatrix}, \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\} \\ L_{01} &= \left\{ \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\} \\ L_{10} &= \left\{ \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{pmatrix} \right\} \\ L_{11} &= \left\{ \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ -1 & \cdot & \cdot \end{pmatrix} \right\} \end{aligned} \tag{8.9}$$

with L_{10} and L_{11} the odd subspaces, and κ as in (5.9). Combining κ with the 16 entries of table 1, we get in all but two cases a non-trivial contraction of osp(2, 1). All 14 of them different, 6 being discrete. They are listed in (5.10) and (5.11). Combining κ with the entries in table 2, we get another 21 contractions of (8.9).

Example 4. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading splits A_1 into three subspaces L_b, L_c, L_d generated by the Pauli matrices. The grading structure and contractions are given in (5.12) and (5.13). The three subspaces are conjugate under the action of the SL(2, $\mathbb{C}$) group, therefore the three contractions in each row of (5.13) give isomorphic algebras. Moreover, the contracted algebras turn out to be isomorphic to the continuous limits of the $\mathbb{Z}_2$- and $\mathbb{Z}_3$-gradings of A_1. The correspondences are easily established, for example, by considering the equality of dimensions of the derived algebras of the contractions.

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